Principally polarized squares of elliptic curves with field of moduli equal to \mathbb{Q}

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Proposition

- There exist exactly 46 genus-2 curves over Q
 with field of moduli Q whose Jacobians are isomorphic to the square of an elliptic curve with complex multiplication by a maximal order.
- Among these 46 curves exactly 13 can be defined over \mathbb{Q} .

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$$\begin{aligned} \mathsf{CM-theory} &\implies E^{\sigma} \simeq E/I_{\sigma} & \text{ for } I_{\sigma} \in \operatorname{Cl}(\mathscr{O}) \\ & \mathsf{Kani} \ (2011) &\implies E^2 \simeq (E/I_{\sigma})^2 &\iff I_{\sigma}^2 = [\mathscr{O}] \end{aligned}$$

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Proposition

A necessary condition for the field of moduli **M** to be contained in \mathbb{K} is that the class group of \mathcal{O} has exponent at most 2.

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Fact

Assuming the Generalized Riemann Hypothesis, there exist 65 fundamental discriminants whose class group is of exponent at most 2.

$\#\operatorname{Cl}(\mathcal{O})$	Discriminants Δ
20	-3, -4, -7, -8, -11, -19, -43, -67, -163
2^1	-15, -20, -24, -35, -40, -51, -52, -88, -91, -115, -123, -148, -187, -232, -235, -267, -403, -427
2 ²	-84, -120, -132, -168, -195, -228, -280, -312, -340, -372, -408, -435, -483, -520, -532, -555, -595, -627, -708, -715, -760, -795, -1012, -1435
2 ³	-420, -660, -840, -1092, -1155, -1320, -1380, -1428, -1540, -1848, -1995, -3003, -3315
2^4	-5460

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Proposition

In genus 2, (E^2, φ) is a Jacobian $\iff \varphi$ is not decomposable $\iff M$ is not congruent to a diagonal matrix.

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 \longrightarrow a matrix M with small coefficients

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Find the polarizations

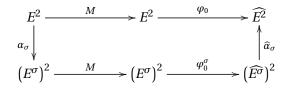
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Fact

For the 65 possible orders, there exist 1226 indecomposable principal polarizations.

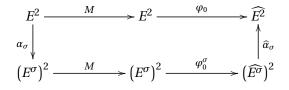
Conditions on (E^2, φ)

• $\mathbf{M} \subseteq \mathbb{K}$ is field of moduli $\iff \forall \sigma \in \operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{K}), (E^2, \varphi) \simeq (E^2, \varphi)^{\sigma},$ *i.e.*, the following diagram commutes



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• In terms of ideals, if $E^{\sigma} \simeq E/I_{\sigma}$ with $I_{\sigma} \in Cl(\mathcal{O})$ and $\mathfrak{a}_{\sigma} \in I_{\sigma}$, then $\forall \sigma \in Gal(\overline{\mathbb{Q}}/\mathbb{K}), \exists P \in GL_2(\mathfrak{a}_{\sigma})$ such that $(n = Norm(\mathfrak{a}_{\sigma}))$

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- Hence Q must be of the form $\begin{pmatrix} x & y \\ z & t \end{pmatrix}$ with x, y, z, $t \in \mathbb{K}$ satisfying

 $\operatorname{Norm}(x) + \operatorname{Norm}(z) = \operatorname{Norm}(y) + \operatorname{Norm}(t) = n \quad \text{and} \quad \overline{x}y + \overline{z}t = 0.$

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$$P = L^{-1}QL = \begin{pmatrix} x - bz & \frac{bx + y - b^2 z - bt}{a} \\ az & bz + t \end{pmatrix} \in M_2(\mathfrak{a}_{\sigma}).$$

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Fact

Among the 1226 Jacobians of genus-2 curves identified earlier, 46 have their field of moduli equal to \mathbb{Q} .

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- Compute an approximation of the Cardona-Quer invariants
- Recognize them as rationals (special form for denominators)

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Proof

For these 13 curves, we have proven that the invariants are correct by having computed the endomorphism ring.

Costa-Mascot-Sijsling-Voight (2017)

Thank you